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THE BUCKLING OF INTERBRACED COLUMNS

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Abstract—The paper presents an analysis of a system of parallel structural columns interlinked by elastic braces, and has the particular aim of determining critical or buckling load levels. The analysis reduces the problem to the solution of a pair of simultaneous difference (recurrence) equations. The nature of this solution is investigated and buckling conditions established. Charts are presented which cover the basic solutions for different numbers and stiffness of braces and for different numbers of columns.

1. INTRODUCTION

In a recent paper on the structural design of column bracing Medland[1] considers critical (buckling) loads for sets of interbraced compression members. The numerical procedure used is a stiffness matrix formulation which incorporates the weakening effect of axial compression on the column stiffnesses[2–4]. The structures analysed include a class consisting of a set of parallel columns which have uniform cross-sectional properties and axial compressive force throughout their length and are inter-connected by elastic spring braces of equal stiffness, spaced so as to divide the length into equal parts (Fig. 1). Only buckling in the plane of the structure is considered. Each column is simply supported at both ends.

The most common example of such a structure is the set of compression chords of a parallel truss roofing system. Bracing against chord buckling in the roof plane is provided by purlins which interlink the chords and normally extend to a cross-braced bay which acts as an effective anchorage. The chords are usually the same size and the purlins are placed at regular intervals within the length. The chords are flexible and the assumed end rotational support condition is relatively unimportant. Simple support is conservative. Reference [1] establishes that the assumption of uniform axial load within the column length is valid and not grossly conservative despite the "parabolic" variation which is actually present.

This paper presents an alternative to the numerical approach[1]. The same basic structure type is considered by means of continuity criteria and the use of difference equations in a manner similar to that presented by Miles[5] for a vibrating beam.

The use of difference equations in this context is certainly not new (e.g. Bleich[6]) but

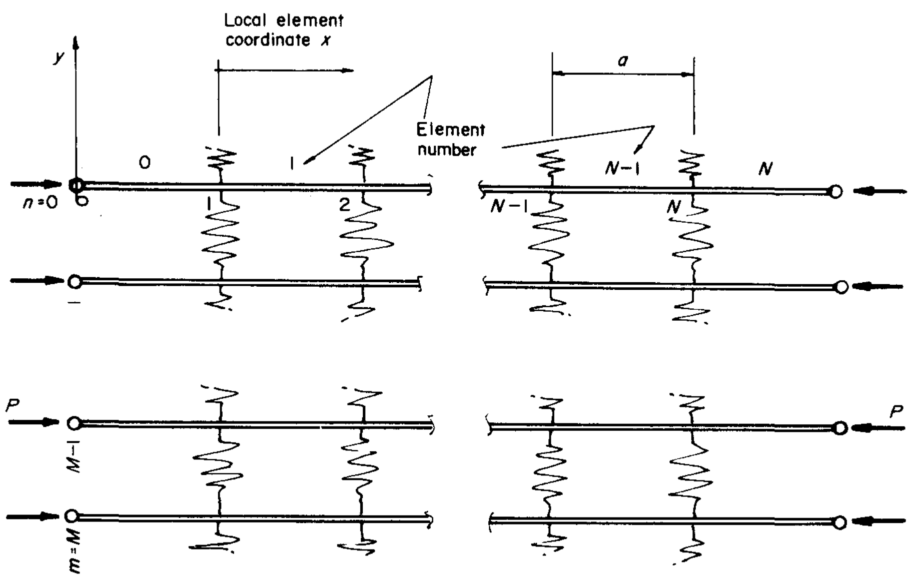

Fig. 1. Multi-column structure system.

the method, which exploits to the full the repetitive nature of the structure, deserves to be more widely used. The case of a single column has also been treated by Budiansky, Seide and Weinberger[7], who used a variational technique which lead to a buckling condition expressed as an infinite series which they had the mathematical insight to express in closed form. The extension to a set of parallel columns is new.

2. GOVERNING DIFFERENCE EQUATIONS

The structure considered is shown in Fig. 1. The same uniform axial compressive load P is applied to each column. The springs all have the same linear elastic stiffness K and are attached at equally spaced nodes numbered $n = 1, N$. The M columns are identified by the variable m with $m = 0$ and $m = M + 1$ being "foundation columns". The general column inter-brace element, n, lies between nodes n and $n + 1$ on column m.

With the sign convention shown in Fig. 2 the bending moment, $\mathcal{M}$, and shear force, S, within an element are expressed in terms of lateral deflection, y, by eqns (1) and (2) respectively.

$$\mathcal{M} = EI\, y'', \tag{1}$$

$$S = \mathcal{M}' + P\, y' = EI\, y''' + Py', \tag{2}$$

where E is Young's modulus and I the second moment of area relevant to lateral bending. The differential equation governing equilibrium within an element is

$$EI\, y^{iv} + P\, y'' = 0, \tag{3}$$

or

$$y^{iv} + \lambda^2\, y'' = 0, \tag{4}$$

where

$$\lambda^2 = P/(EI). \tag{5}$$

The simply supported end conditions mean that both $\mathcal{M}$ and y are zero at the extreme nodes $n = 0$ and $n = N + 1$ of each column. Across each internal node there must be continuity of deflection, slope and bending moment, while a discontinuity of shearing force must develop to accommodate the spring reaction at that node.

An appropriate form for the general solution of eqn (4) within the element n, m is

$$y_{n,m} = A_{n,m}[a \sin \lambda x - x \sin \lambda a]/a + B_{n,m}[a \sin\lambda (a - x) - (a - x) \sin \lambda a]/a + C_{n,m}\, x/a + D_{n,m}\, (a - x)/a, \tag{6}$$

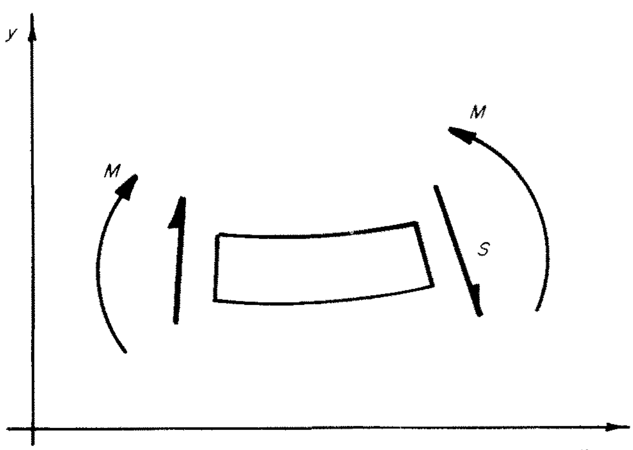

Fig. 2. Sign convention for Bending Moment and Shearing Forces.

where x is a local coordinate for which $x = 0$ at node n,m and $x = a$ at node $n + 1,m$. The terms containing $A_{n,m}$ and $B_{n,m}$ in eqn (6) vanish at $x = 0$ and $x = a$. Thus the deflection at node n, m is $D_{n,m}$ and that at node $n + 1,m$ is $C_{n,m}$. Continuity of displacement at node $n + 1,m$ is therefore expressed by

$$C_{n,m} = D_{n+1,m}. \tag{7}$$

The terms containing $C_{n,m}$ and $D_{n,m}$ in eqn (6) are linear in x and do not contribute to the bending moment $\mathcal{M}$ (eqn 1). Continuity of $\mathcal{M}$ across node $n + 1,m$ is expressed by

$$\lambda^2 EI \sin \lambda a\, A_{n,m} = \lambda^2 EI \sin \lambda a\, B_{n+1,m}. \tag{8}$$

Thus, either

$$\sin \lambda a = 0, \tag{9}$$

or

$$A_{n,m} = B_{n+1,m}. \tag{10}$$

Eqn (9) corresponds to the axial load having reached the Euler buckling load of the inter-nodal length a, a load at which the structure is capable of buckling without extending the springs. Provided P is less than that value, eqn (10) must hold.

Continuity of slope at node n,m is governed by eqn (11) which results from equating $y'_{n,m}(0)$ to $y'_{n-1,m}(a)$ through eqn (6), together with the replacement of C and A parameters in terms D and B using eqns (7) and (10).

$$B_{n+1,m}[t - \sin t] - 2B_{n,m}[t \cos t - \sin t] + B_{n-1,m}[t - \sin t] + D_{n+1,m} - 2D_{n,m} + D_{n-1,m} = 0, \tag{11}$$

where

$$t = \lambda a. \tag{12}$$

The shearing force, $S_{n,m}$, within element n,m can be obtained from eqn (2) and derivatives of eqn (6) and again making use of eqns (7) and (10),

$$S_{n,m} = \frac{\lambda^2 EI}{a}[(B_{n,m} - B_{n+1,m}) \sin t + D_{n+1,m} - D_{n,m}]. \tag{13}$$

Across the node n,m the shearing forces in elements $n - 1,m$ and n,m must balance the spring reaction to the deflection of node n,m relative to its corresponding nodes on columns $m - 1$ and $m + 1$. With the sign convention of Fig. 2, this balance is expressed as,

$$S_{n-1,m} - S_{n,m} = K(D_{n,m} - D_{n,m-1}) + K(D_{n,m} - D_{n,m+1}). \tag{14}$$

Equations (13) and (14) give a second relationship between the B and D parameters, namely,

$$\sin t[B_{n+1,m} - 2B_{n,m} + B_{n-1,m}] - [D_{n+1,m} - 2D_{n,m} + D_{n-1,m}] = \frac{Ka^3}{t^2 EI}[-D_{n,m+1} + 2D_{n,m} - D_{n,m-1}]. \tag{15}$$

3. SOLUTION OF THE DIFFERENCE EQUATIONS

Equations (11) and (15) are a pair of linear homogeneous difference equations for the $B_{n,m}$ and $D_{n,m}$ parameters. If attention is focussed first on the variation with n it can be seen that a possible form for the solution of the pair is

$$B_{n,m} = B_m \left(\begin{matrix}\sin\\ \cos\end{matrix}\, n\theta\right), \quad D_{n,m} = D_m \left(\begin{matrix}\sin\\ \cos\end{matrix}\, n\theta\right). \tag{16}$$

Substitution into eqns (11) and (15), and the subsequent cancellation of the common factors sin $n\theta$ and cos $n\theta$ lead to

$$B_m[(1-\cos\theta)(t-\sin t)-t(1-\cos t)]+D_m(1-\cos\theta)=0 \tag{17}$$

and

$$\sin t(1-\cos\theta)B_m-(1-\cos\theta)D_m=\frac{\beta^*}{t^2}[-D_{m+1}+2D_m-D_{m-1}], \tag{18}$$

where

$$\beta^*=Ka^3/(2\,EI). \tag{19}$$

The fact that each column is simply supported at the end node $n=0$ requires that

$$B_{0,m}=D_{0,m}=0 \tag{20}$$

and this precludes the cosine form in this problem.

The term on the right hand side of eqn (18) suggests the substitution of the further trial form

$$B_m=B\left({}^{\sin}_{\cos}\,m\alpha\right),\quad D_m=D\left({}^{\sin}_{\cos}\,m\alpha\right). \tag{21}$$

Again the cos $m\alpha$ components may be eliminated because the column $m=0$, the "foundation column", remains straight. Thus

$$B_0=D_0=0. \tag{22}$$

Equation (17) becomes

$$B[(1-\cos\theta)(t-\sin t)-t(1-\cos t)]+D(1-\cos\theta)=0, \tag{23}$$

and eqn (18) takes the form

$$B\sin t(1-\cos\theta)+D\left[\frac{2\beta^*(1-\cos\alpha)}{t^2}-(1-\cos\theta)\right]=0. \tag{24}$$

It can be seen from eqn (24) that the factor

$$\beta=2\beta^*(1-\cos\alpha) \tag{25}$$

becomes the effective stiffness of the set of springs in series. The basic brace stiffness β^* is scaled by $2(1-\cos\alpha)$. If the "column" $M+1$ is also a rigid foundation it is clear that $B_{M+1}=D_{M+1}=0$ and thus

$$\sin(M+1)\alpha=0, \tag{26}$$

i.e.

$$\alpha=k\pi/(M+1)\text{ for }k=1,M\dagger. \tag{27}$$

†A more general foundation condition for column M is provided by using a spring of stiffness μK between M and $M+1$. This leads to the equation

$$\sin(M+1)\alpha=(1-\mu)\sin M\alpha,$$

of which eqn (26) is a special case. The $\mu=0$ case, i.e. no spring outside column M, results in

$$\alpha=k\pi/(2M+1)\text{ for }k=1,3,\ldots(2M-1).$$

In order that non-trivial solutions for B and D can exist, the 2×2 determinant of their coefficients in eqns (23) and (24) must be zero. Thus

$$[(1-\cos\theta)(t-\sin t)-t(1-\cos t)]\,[\beta/t^2-(1-\cos\theta)]=\sin t\,(1-\cos\theta)^2. \tag{28}$$

Equation (28) is a quadratic in $(1-\cos\theta)$ and, provided† that the roots lie between 0 and 2, two values of θ can be found for a given pair of values of t and β, i.e. for a given axial load, spring stiffness and foundation condition for column M. These two values of θ lead to particular solution pairs of the form:

$$\begin{aligned} B_{n,m} &= B^{(1)}\sin n\theta_1 \\ D_{n,m} &= D^{(1)}\sin n\theta_1 \end{aligned} \quad \text{and} \quad \begin{aligned} B_{n,m} &= B^{(2)}\sin n\theta_2 \\ D_{n,m} &= D^{(2)}\sin n\theta_2. \end{aligned} \tag{29}$$

A linear combination of these two solution pairs provides the general solution to the difference eqns (11) and (15) in the cases where all columns have node $n=0$ simply supported.

4. BUCKLING OF THE STRUCTURE

The general solution form derived from eqn (29) is

$$\begin{aligned} B_{n,m} &= B^{(1)}\sin n\theta_1 + B^{(2)}\sin n\theta_2 \\ D_{n,m} &= r_1\,D^{(1)}\sin n\theta_1 + r_2\,D^{(2)}\sin n\theta_2 \end{aligned} \tag{30}$$

where r_1 and r_2 are the ratios of D/B as obtained from eqn (23) or (24).

At node $N+1$,

$$\begin{aligned} B^{(1)}\sin(N+1)\theta_1 + B^{(2)}\sin(N+1)\theta_2 &= 0 \\ r_1 B^{(1)}\sin(N+1)\theta_1 + r_1 B^{(2)}\sin(N+1)\theta_2 &= 0, \end{aligned} \tag{31}$$

and if a non-zero solution for $B^{(1)}$ and $B^{(2)}$ is to be found (i.e. if a buckled form is possible), then

$$\sin(N+1)\theta_1 = 0 \text{ or } \sin(N+1)\theta_2 = 0. \tag{32}$$

Thus,

$$\theta_1,\quad \theta_2 = j\pi/(N+1),\qquad j=1,N. \tag{33}$$

These values of θ must be used in the buckling criterion, eqn (28), which upon rearrangement and the substitution of

$$X = 1-\cos(j\pi/(N+1)), \tag{34}$$

becomes

$$X^2 - X[(1-\cos t)+\beta(t-\sin t)/t^3] + \beta\,(1-\cos t)/t^2 = 0. \tag{35}$$

In order to determine the lowest critical level of axial load (measured by the non-dimensional factor t) for a given structure (N, β), several column buckling mode shapes (j) must be considered. It is clear from eqn (35) that the effect of there being m columns is confined to modifying the basic non-dimensional spring stiffness parameter β^* to β.

5. THE BUCKLING CONDITIONS

The form of the relationship between the critical t value and β, N and j is discussed with reference to specific cases. The relationships are summarised on Fig. 3.

†A detailed examination of this proviso is presented in the Appendix.

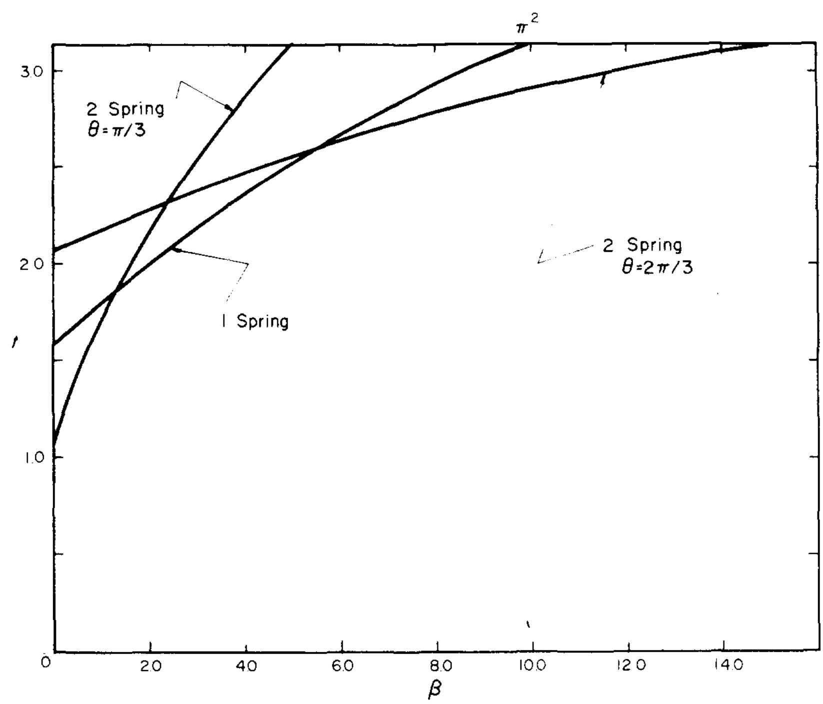

Fig. 3. One and two spring $\beta:t$ relationships.

One spring

When $N = 1$, i.e. when only the mid-point of each column is elastically restrained, the only possible buckling load below the limiting case $t = \pi$ (discussed in reference to eqn 9) is associated with the mode in which $\theta = \pi/2$. For this case eqn (35) becomes (see also Ref. [8])

$$\beta = \frac{t^3}{(t - \tan t)}. \tag{36}$$

For $\beta = 0$, eqn (35) gives $t = \pi/2$ and the buckling mode within each column is that of pure Euler buckling of the full column length $2a$, a mode often descriptively known as C-buckling. With increase in β the critical value of t rises steadily as shown on Fig. 3. At the β value π^2, eqn (36) shows the critical t value to be π. At this load the C-buckling form is still possible but the form known as S-buckling is also possible. S-buckling involves no extension of the spring and is a pure second Euler mode for the full length column. For $\beta > \pi^2$, only S-buckling is possible and the critical t value does not rise above π because the spring is no longer absorbing energy.

Two springs

When $N = 2$, θ can take the values $\pi/3$ and $2\pi/3$.

$$\text{For } \theta = \pi/3, \quad \beta = \frac{t^3(1 - 2\cos t)}{2(t + \sin t - 2t\cos t)}, \tag{37}$$

$$\text{and for } \theta = 2\pi/3, \quad \beta = \frac{3t^3(1 + 2\cos t)}{2(t - 3\sin t + 2t\cos t)}. \tag{38}$$

The curves for these two cases are also shown on Fig. 3. They cross at $\beta = 0.245\,\pi^2$, $t = 0.744\,\pi$. For $\beta \leq 0.24\,\pi^2$, C-buckling occurs, for $0.245\,\pi^2 \leq \beta \leq 3\pi^2/2$ S-buckling is involved and for $\beta \geq 3\pi^2/2$, the column again buckles but at a load independent of β. This last load corresponds to the Euler buckling load of the element length a, and the springs are not extended.

More than two springs

The behaviour when there are N springs is an obvious extension of the previous two cases. For each θ_j, $j = 1, N$ the critical t vs β relationship lies on a curve similar to those shown in Fig. 3. Each joins the point $\beta = 0$, $t = j\pi/(N+1)$ to the point $\beta = (1 - \cos\theta_j)\pi^2$, $t = \pi$. The minimum β value required to prevent the springs being strained (i.e. to raise critical t to π) is given by

$$\beta_T = [1 - \cos(N\pi/(N+1))]\pi^2 = [1 + \cos(\pi/(N+1))]\pi^2. \tag{39}$$

Clearly, as N becomes larger β_T approaches an upper limit of $2\pi^2$.

Enveloping curve

The foregoing suggests that for different N and j, the various curves in the (β, t) plane given by eqn (35) all touch a single enveloping curve. This can be found by considering the one-parameter family of curves given by eqn (35), X being the parameter, whose envelope obtained by the use of standard procedures (as given, say in Ref. [9]) has equation

$$[(1 - \cos t) + \beta(t - \sin t)/t^3]^2 - 4\beta(1 - \cos t)/t^2 = 0. \tag{40}$$

This curve provides a safe (conservative) estimate of the critical load level for a given spring stiffness. Equation (40) can be manipulated into a form,

$$\frac{(t - \sin t)^2}{t^6}(\beta - \beta_1)(\beta - \beta_2) = 0. \tag{41}$$

where

$$\beta_1(t) = \frac{t^3(1 - \cos t)}{(\sqrt{t} + \sqrt{(\sin t)})^2}, \quad \beta_2(t) = \frac{t^3(1 - \cos t)}{(\sqrt{t} - \sqrt{(\sin t)})^2}. \tag{42}$$

The axial load factor t is still limited to the range $0 \leq t \leq \pi$. $\beta_1(t)$ is obviously the lower of the two and hence the graph of $\beta_1(t)$ as a function of t is the appropriate envelope.

If t is replaced by the alternative non-dimensional parameter

$$\rho = t^2/\pi^2 = P/P_E, \tag{43}$$

where

$$P_E = \pi^2 EI/a^2, \tag{44}$$

and if β is replaced by

$$s = \beta/2\pi^2, \tag{45}$$

the plot of the lower envelope ρ vs s is the close approximation to a quadrant of a circle shown on Fig. 4, which provides an extremely concise summary of a very general problem.

6. CONCLUSIONS

In defining, by means of the single eqn (35), the buckling criterion for a structure consisting of any number of simply supported solumns interlinked by any number of equally spaced braces of any stiffness, and in allowing a range of support conditions for the extreme columns, the technique presents an extremely efficient method for the determination of the critical load criterion. The general results obtained are in accord with the particular numerical results given in Ref. [1]. The method of computation used in Ref. [1] was that of assembling the stiffness matrix for the structure and iterating the load factor until the matrix became singular. This provides one point on the plot relating β, N, M, and t and is very inefficient, specially for many

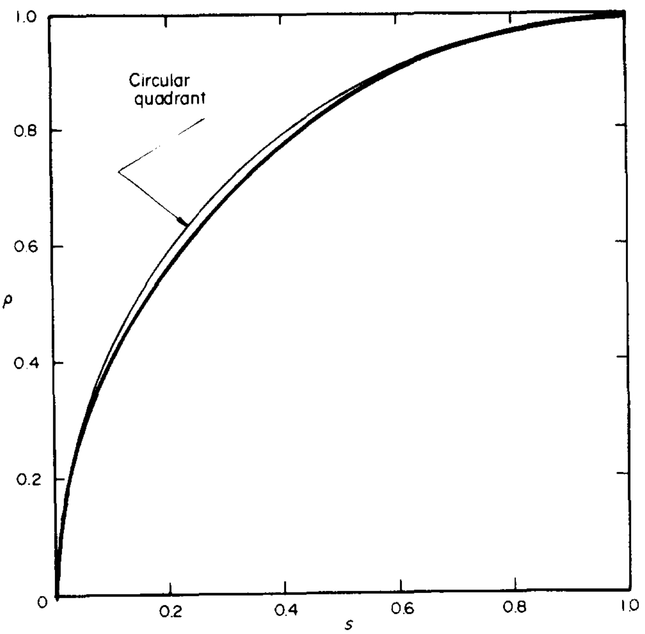

Fig. 4. Lower envelope to the spring stiffness: axial load relationships.

columns, in comparison to the technique described in this paper. The fact that a scaling of β^* allowed the single column curves to be used for multicolumn analysis was noted in Ref. [1], but the scale factor was derived numerically. This paper provides a proper analytical explanation and evaluation of the scaling factor, $2(1-\cos\alpha)$, and presents the formula by which α may be determined for a greater range of support conditions. A development of the scaling factor through a physical argument has been presented by Williams[10].

Curves relating the β and t factors for columns with non-uniform axial load and non-uniform section properties within their length are presented in [1]. The technique of this paper cannot be extended to these cases, nor to less regular systems involving non-simple end support. Reference[1] does, however, show that for many practical applications (e.g. the analysis of truss compression chords) the curves derived from the constant axial load—constant section properties—simple support assumption provide a not too conservative estimate of actual critical loads.

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APPENDIX

Further examination of the solution of the difference equations

It is now necessary to examine the nature of the roots of the quadratic equation in $X = (1-\cos\theta)$, given by eqn (35), namely

$$F(X) = X^2 - X[(1-\cos t) + \beta(t-\sin t)/t^3] + \beta(1-\cos t)/t^2 = 0. \qquad (A1)$$

It is clear that,

$$F(0) = \beta(1 - \cos t)/t^2 > 0, \tag{A2}$$

$$F'(0) = -(1 - \cos t) - \beta(t - \sin t)/t^3 < 0 \tag{A3}$$

$$F(2) = 2(1 + \cos t) + \beta(2 \sin t - t - t \cos t)/t^3 = 2(1 + \cos t) + 4\beta \cos \tfrac{1}{2}t(\sin \tfrac{1}{2}t - \tfrac{1}{2}t \cos \tfrac{1}{2}t)/t^3, \tag{A4}$$

which is certainly positive for $0 \leq t \leq \pi$ (since $\cos\frac{1}{2}t > 0$ and $\tan\frac{1}{2}t > \frac{1}{2}t$ in that range),

$$F'(2) = 3 + \cos t - \beta(t - \sin t)/t^3, \tag{A5}$$

and it is clear that $F'(2) \gtrless 0$ according as

$$\beta \gtrless \gamma(t) = \frac{(3 + \cos t)t^3}{t - \sin t}. \tag{A6}$$

From the above, it follows that if the roots are real, then both lie between $X = 0$ and $X = 2$ or both are greater than 2.

Furthermore, if the real roots lie between 0 and 2 then the point (β, t) must lie to the left of the curve $\beta = \gamma(t)$ and if they are greater than 2 then (β, t) must lie to the right of $\beta = \gamma(t)$.

The minimum value of $F(x)$ is found to be equal to

$$-\frac{1}{4}\left[\beta^2\frac{(t - \sin t)^2}{t^6} - 2\beta(1 - \cos t)\frac{(t + \sin t)}{t^3} + (1 - \cos t)^2\right] = -\frac{(t - \sin t)^2}{4t^6}(\beta - \beta_1)(\beta - \beta_2), \tag{A7}$$

where $\beta_1(t)$ and $\beta_2(t)$ are given in eqn (42).

The graph of $\beta_1(t)$, $\beta_2(t)$ and $\gamma(t)$ are shown in Fig. 5.

It follows that if the point (β, t) lies in region I, as shown, the roots X are real and since this region is to the left of γ, these real roots lie between 0 and 2 and the theory of the previous sections applies.

If (β, t) lies in II both roots are complex. If (β, t) lies in III, a region to the right of γ, both roots are real but greater than 2.

If eqn (35) has complex roots, then $\cos \theta$ is complex and it follows that the simple solution pair,

$$B_n = \sin n\theta, \quad D_n = r \sin n\theta \tag{A8}$$

where r is given by the D/B ratio calculated from eqn (23) or (24), must be modified, for now both $\theta = p + iq$ and $r = u + iv$ are complex. Furthermore, since another simple solution pair can be obtained by taking the complex conjugate of eqn (A8) it becomes clear that solutions can be obtained by taking the real and imaginary parts of the above solution pair.

In this manner, the following can be obtained:

$$\begin{aligned} B_{n,m} &= B^{(1)} \sin np \cosh nq + B^{(2)} \cos np \sinh nq, \\ D_{n,m} &= B^{(1)}[u \sin np \cosh nq - v \cos np \sinh nq] + B^{(2)}[u \cos np \sinh nq + v \sin np \cosh nq]. \end{aligned} \tag{A9}$$

It should be noted that in eqn (A9) $B_{n,m}$ and $D_{n,m}$ both vanish for $n = 0$. However, with the possibility of buckling in mind, to insist that both $B_{N+1,m}$ and $D_{N+1,m}$ vanish for non-trivial values of $B^{(1)}$ and $B^{(2)}$ would require the vanishing of the 2×2 coefficient determinant, which here leads to the condition,

$$v[\cosh^2(N + 1)q - \cos^2(N + 1)p] = 0,$$

which is impossible.

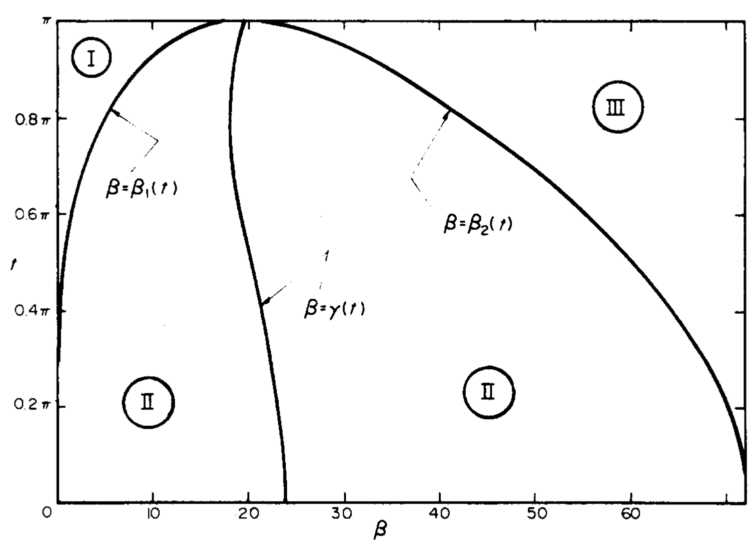

Fig. 5. Plots of $\beta_1(t)$, $\beta(t)$ and $\gamma(t)$.

Thus there can be no buckling if the point (β, t) lies in region II.

If (β, t) lies in region III, the two values of X are real and hence so also is r. But as $X > 2$, each value of $\cos\theta$ must be less than -1 and this implies that θ is complex and of the form,

$$\theta_1 = \pi + iq_1 \qquad \theta_2 = \pi + iq_2. \tag{A10}$$

The appropriate solution pair is now

$$\begin{aligned} B_{n,m} &= (-1)^n[B^{(1)}\sinh nq_1 + B^{(2)}\sinh nq_2] \\ D_{n,m} &= (-1)^n[rB^{(1)}\sinh nq_1 + rB^{(2)}\sinh nq_2], \end{aligned} \tag{11}$$

from which it can also be shown that buckling cannot take place.